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Nonvanishing modulo ℓ of Fourier coefficients of Jacobi forms

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Abstract

Let $\phi = \sum_{r^2 \leq 4mn} c(n, r) q^n \zeta^r$ be a Jacobi form of weight k (with $k > 2$ if ϕ is not a cusp form) and index m with integral algebraic coefficients which is an eigenfunction of all Hecke operators T_p , $(p, m) = 1$, and which has at least one nonvanishing coefficient $c(n^*, r^*)$ with r^* prime to m . We prove that for almost all primes ℓ there are infinitely many fundamental discriminants $D = r^2 - 4mn < 0$ prime to m with $v_\ell(c(n, r)) = 0$, where v_ℓ denotes a continuation of the ℓ -adic valuation on \mathbb{Q} to an algebraic closure. As applications we show indivisibility results for special values of Dirichlet L -series and for the central critical values of twisted L -functions of even weight newforms.

Keywords: Nonvanishing, Indivisibility, Fourier coefficients, Jacobi forms, Special values of L -functions

1 Introduction and main results

Let k and m be positive integers. Ono and Skinner [11] and Bruinier [1] have shown, using different methods, that for a non-trivial Hecke eigenform $f = \sum_{n \geq 0} a(n) q^n$ of weight $k + 1/2$ for $\Gamma_0(4m)$ with integral algebraic coefficients, which is not a linear combination of Shimura theta functions, almost all primes ℓ have the property that there are infinitely many square free n with $v_\ell(a(n)) = 0$, where v_ℓ denotes a continuation of the ℓ -adic valuation on \mathbb{Q} to a fixed algebraic closure.

In this work, we adapt the methods from [1] to prove an analogous result for the Fourier coefficients of Jacobi forms:

Theorem 1.1. *Let $\phi \in J_{k,m}$ be a Jacobi form of weight k (with $k > 2$ if ϕ is not a cusp form) and index m with integral algebraic coefficients $c(n, r)$ which is an eigenform of all Hecke operators T_p for $(p, m) = 1$. Suppose that there is a discriminant $D^* = r^{*2} - 4mn^* < 0$ with $(D^*, m) = 1$ and $c(n^*, r^*) \neq 0$. For almost all primes ℓ with $(\ell, 2m) = 1$ there exist infinitely many fundamental discriminants $D = r^2 - 4mn < 0$ prime to m such that $v_\ell(c(n, r)) = 0$.*

The finite set of primes ℓ for which the theorem does not hold can be described explicitly in terms of the Hecke eigenvalues λ_p of ϕ , see (5) and (6) below. The ‘bad’ primes are essentially those dividing $\lambda_p + (p^{k-1} + p^{k-2})$ or $\lambda_p - (p^{k-1} + p^{k-2})$ for all p . For cusp forms, the proof that there are only finitely many of these primes relies on

the Shimura correspondence and the elementary estimate $|\lambda_p| < p^{k-1} + p^{k-2}$ for the eigenvalues of integral weight cusp forms from [8].

Further, if $S = \{p_1, \dots, p_s\}$ is a finite set of odd primes which are coprime to m and if $\varepsilon_1, \dots, \varepsilon_s \in \{\pm 1\}$, then we can find (for almost all primes ℓ) infinitely many fundamental discriminants $D < 0$ as in the theorem satisfying the additional local conditions $\left(\frac{D}{p_j}\right) = \varepsilon_j$ for all $p_j \in S$. For a more detailed version of the theorem, which also covers Jacobi forms of higher level, see Theorem 3.5.

The condition $c(n^*, r^*) \neq 0$ for some $D^* = r^{*2} - 4mn^* < 0$ prime to m is obviously necessary, and is violated exactly by the forms in $\sum_{d^2|m, d>1} J_{k,m/d^2} U_d$ where $\phi|U_d(\tau, z) = \phi(\tau, dz)$, compare Lemma 3.1 in [15]. In particular, the theorem is applicable for newforms. Further, for a newform ϕ we can replace the requirement of a special discriminant D^* as in the theorem by a somewhat nicer looking condition on the Atkin-Lehner eigenvalues of ϕ , see Proposition 3.6.

The proof of the theorem goes along similar lines as the proof of the corresponding Theorem 4 from [1] for half-integral weight forms. The main ingredients are commutation relations of Hecke operators, Fricke involutions (with regard to the level), and twisting and projection operators for Jacobi forms on the group $\Gamma_0(N^2) \ltimes (N\mathbb{Z} \times \mathbb{Z})$, together with the q -expansion principle for integral weight modular forms. However, the aforementioned operators for Jacobi forms of higher level seem to be not very common, so it is interesting to see that similar arguments as for half-integral weight modular forms also work for Jacobi forms.

An important tool in the proof of Theorem 1.1 is the following result on the algebraicity of Fourier expansions of a Jacobi form at the cusps, which might be of independent interest.

Theorem 1.2. *Let $\phi \in J_{k,m}(\Gamma_1(N^2) \ltimes (N\mathbb{Z} \times \mathbb{Z}))$ have integral algebraic Fourier coefficients $c(n, r)$ and let ℓ be a prime with $(\ell, 2mN) = 1$. For $M \in SL_2(\mathbb{Z})$ the function $\phi|_{k,m}M$ has algebraic Fourier coefficients and it holds $v_\ell(\phi|_{k,m}M) \geq v_\ell(\phi)$.*

Here we wrote $v_\ell(\phi) = \inf_{n,r} v_\ell(c(n, r))$ for a Jacobi form $\phi = \sum_{r^2 \leq 4mn} c(n, r) q^n \zeta^r$. For the proof, we apply the q -expansion principle to the components of the vector valued modular form corresponding to ϕ via its theta decomposition.

Let us now give two applications of Theorem 1.1, both of which are strengthenings of applications from [1]: First, we consider the central critical values of twisted L -functions of even weight newforms. By the results of [15], the space of cuspidal Jacobi newforms $J_{k+1,m}^{\text{cusp,new}}$ of weight $k+1$ and index m is isomorphic as a module over the Hecke algebra to the space $S_{2k}^{\text{new,-}}(m)$ of elliptic newforms of weight $2k$ and level m whose L -function $L(f, s)$ has a minus sign in its functional equation. Let $f \in S_{2k}^{\text{new,-}}(m)$ be a normalised newform and let $\phi \in J_{k+1,m}^{\text{cusp,new}}$ be the corresponding Jacobi newform with integral algebraic coefficients. If $D = r^2 - 4mn < 0$ with $(D, m) = 1$ is a fundamental discriminant which is a square mod $4m$, the formula of Waldspurger [16] in the explicit form given by Gross, Kohnen and Zagier [6] states that

$$\frac{|c(n, r)|^2}{\langle \phi, \phi \rangle} = \frac{(k-1)!}{2^{2k-1} \pi^k m^{k-1}} |D|^{k-1/2} \frac{L(f, D, k)}{\langle f, f \rangle}, \quad (1)$$

where $L(f, D, s)$ denotes the L -function of f twisted by the Kronecker symbol $\left(\frac{D}{\cdot}\right)$. Now Theorem 1.1 implies the following result.

Corollary 1.3. *For a normalised newform $f \in S_{2k}^{\text{new},-}(m)$ there exists a nonzero complex number Ω_f such that for almost all primes ℓ with $(\ell, 2m) = 1$ there are infinitely many fundamental discriminants $D < 0$ prime to m which are squares mod $4m$ such that*

$$v_\ell \left(\frac{L(f, D, k) |D|^{k-1/2}}{\Omega_f} \right) = 0.$$

Note that for a newform $f \in S_{2k}^{\text{new},+}(m)$ the functional equation of the twisted L -function implies $L(f, D, k) = 0$ for fundamental discriminants $D < 0$ which are coprime to m and squares mod $4m$. Compared to the very similar Theorem 5 in [1] or Corollary 2 in [11], the fundamental discriminants D in our corollary are now all coprime to m and squares mod $4m$.

As explained in Corollaries 3 and 4 in [11], a result like Corollary 1.3 has implications for the orders of the Tate-Shafarevich groups of quadratic twists of elliptic curves over \mathbb{Q} . For related results, see also [9].

As a second application we consider the coefficients of the Eisenstein series $E_{k,1} \in J_{k,1}$ for even $k \geq 4$, which are given by special values of Dirichlet L -series. $E_{k,1}$ is an eigenform of all Hecke operators T_p with eigenvalue $\sigma_{2k-3}(p) = 1 + p^{2k-3}$. By Theorem 2.1 in [5], the Fourier coefficients $e_{k,1}(n, r)$ of $E_{k,1}$ for fundamental discriminants $D = r^2 - 4n < 0$ are given by

$$e_{k,1}(n, r) = \frac{L(2-k, \chi_D)}{\zeta(3-2k)},$$

where $\chi_D = \left(\frac{D}{\cdot}\right)$. It is well known that $L(2-k, \chi_D) = -B_{k-1, \chi_D}/(k-1)$ with the generalised Bernoulli numbers B_{k-1, χ_D} , so the values $L(2-k, \chi_D)$ are non-zero rational numbers whose denominators are divisible only by primes ℓ such that $(\ell-1)$ divides $2(k-1)$, see [3]. Hence the coefficients of $\zeta(3-2k)E_{k,1}$ are ℓ -integral rational numbers with bounded denominators for almost all primes ℓ . The same is true for the coefficients of $\zeta(3-2k)E_{k,1}|V_m$, where $V_m : J_{k,1} \rightarrow J_{k,m}$ is the usual Hecke-type operator defined in [5], §4. Moreover, for any $\phi \in J_{k,1}$ and a fundamental discriminant $D = r^2 - 4mn < 0$ we have $c_{\phi|V_m}(n, r) = c_{\phi}(mn, r)$ by the explicit formula for the action of V_m on the Fourier expansion of ϕ . Thus Theorem 1.1 applied to a suitable multiple of $\zeta(3-2k)E_{k,1}|V_m$ gives the following result.

Corollary 1.4. *Let m be a positive integer and let $k \geq 4$ be an even integer. For almost all primes ℓ with $(\ell, 2m) = 1$ there are infinitely many fundamental discriminants $D < 0$ prime to m which are squares mod $4m$ such that*

$$v_\ell(L(2-k, \chi_D)) = 0.$$

On the other hand, for odd k and negative D the functional equation of $L(s, \chi_D)$ gives $L(2-k, \chi_D) = 0$.

Eventually, we remark that the techniques of this paper can be used to prove analogous results for skew-holomorphic Jacobi forms as defined in [14]: Recall that a skew-holomorphic Jacobi form of weight k and index m is a smooth function

$\phi : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$ which is holomorphic in z , which is invariant under a modified slash action $|_{k,m}^*$ of $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ where $(c\tau + d)^k$ is replaced by $(c\bar{\tau} + d)^{k-1}|c\tau + d|$ (see below for the slash operator $|_{k,m}$ on holomorphic Jacobi forms), and which has a Fourier expansions of the form

$$\phi(\tau, z) = \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 \geq 4mn}} c(n, r) e\left(\frac{r^2 - 4mn}{2m} i\tau\right) q^n \zeta^r, \quad (\tau = u + iv).$$

If we replace $r^2 \leq 4mn$ by $r^2 \geq 4mn$ and $|_{k,m}$ by $|_{k,m}^*$, and add a factor $e\left(\frac{r^2 - 4mn}{2m} i\tau\right)$ in the Fourier expansions in all proofs in Section 2, we see that the results in Section 2 are also valid for skew-holomorphic forms.

Since a skew-holomorphic Jacobi form also has a theta decomposition and thereby corresponds to a vector valued modular form (see [2], Section 1.1), the proof of Theorem 1.2 also works for skew-holomorphic forms.

Further, Theorem 3.5 and the propositions and lemmas used for its derivation have counterparts for skew-holomorphic Jacobi forms (just replace $D \leq 0$ with $D \geq 0$ everywhere) and can be proved in the same way. But we have to be careful with the remarks after Theorem 3.5: It seems (to the best knowledge of the author) that the analog of the isomorphism between $J_{k+1,m}^{\text{cusp}}$ with a subspace of S_{2k} for skew-holomorphic Jacobi forms has not been proved in the literature, so estimates for the eigenvalues of skew-holomorphic Jacobi forms are not readily available.

The situation is similar with the Waldspurger formula (1). Although it should be true for skew-holomorphic forms by similar arguments as in the holomorphic case, we could not find a proof in the literature. Assuming the formula, we get an analog of Corollary 1.3 for newforms $f \in S_{2k}^{\text{new},+}(m)$, now yielding infinitely many positive fundamental discriminants $D = r^2 - 4mn > 0$ with $v_\ell(L(f, D, k)|D|^{k-1/2}/\Omega_f) = 0$.

2 Operators on Jacobi forms of higher level

For the basic facts about Jacobi forms (of level 1) we refer the reader to the standard reference [5]. In this section we define Hecke operators and the Fricke involution on Jacobi forms for $\Gamma_0(N^2) \ltimes (N\mathbb{Z} \times \mathbb{Z})$, as well as certain twisting and projection operators, and we study their commutation relations.

Let $e(x) := e^{2\pi i x}$ and $e^m(x) := e^{2\pi i m x}$ for $x \in \mathbb{C}$ and $m \in \mathbb{Z}$. The real Jacobi group $SL_2(\mathbb{R}) \ltimes \mathbb{R}^2 \cdot S^1$ with group law

$$[M, X, \xi] \cdot [M', X', \xi'] = \left[MM', XM' + X', \xi \xi' e\left(\det\begin{pmatrix} XM' \\ X' \end{pmatrix}\right) \right]$$

acts on holomorphic functions $\phi : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$ by

$$\begin{aligned} \phi \Big|_{k,m} \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, [\lambda, \mu], \xi \right] (\tau, z) \\ = \xi^m (c\tau + d)^{-k} e^m \left(-\frac{c(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2\tau + 2\lambda z + \lambda\mu \right) \\ \times \phi \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right). \end{aligned}$$

To simplify the notation we sometimes write $\phi|_{k,m}M$ for $\phi|_{k,m}[M, 0, 1]$ and $\phi|_mX$ for $\phi|_{k,m}[1, X, 1]$.

Definition 2.1. Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a subgroup of finite index, let χ be a character of Γ and let $L \subseteq \mathbb{R}^2$ be a rank 2 lattice which is invariant under right multiplication by Γ . A Jacobi form of weight k and index m for $\Gamma \ltimes L$ with character χ is a holomorphic function $\phi : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{C}$ satisfying the following properties:

- 1 $\phi|_{k,m}M = \chi(M)\phi$ for every $M \in \Gamma$.
- 2 $\phi|_mX = \phi$ for every $X \in L$.
- 3 For every $M \in \mathrm{SL}_2(\mathbb{Z})$, the function $\phi|_{k,m}M$ has a Fourier expansion of the form

$$\sum_{\substack{n,r \in \mathbb{Q} \\ r^2 \leq 4mn}} c_M(n,r) q^n \zeta^r, \quad (q^n = e(n\tau), \zeta^r = e(rz)),$$

with coefficients $c_M(n,r) \in \mathbb{C}$.

If for all $M \in \mathrm{SL}_2(\mathbb{Z})$ we have $c_M(n,r) = 0$ whenever $r^2 - 4mn = 0$, we call ϕ a Jacobi cusp form. The corresponding spaces of Jacobi forms will be denoted by $J_{k,m}(\Gamma \ltimes L, \chi)$ and $J_{k,m}^{\mathrm{cusp}}(\Gamma \ltimes L, \chi)$, and for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $L = \mathbb{Z}^2$ we just write $J_{k,m}$ and $J_{k,m}^{\mathrm{cusp}}$.

In the following, we will mainly work with Jacobi forms for the group $\Gamma_0(N^2) \ltimes (N\mathbb{Z} \times \mathbb{Z})$ and character $\chi(M) = \chi(d)$ for $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N^2)$, where N is a positive integer and χ is a Dirichlet character mod N^2 . Note that the transformation behaviour of $\phi \in J_{k,m}(\Gamma_0(N^2) \ltimes (N\mathbb{Z} \times \mathbb{Z}), \chi)$ under $N\mathbb{Z} \times \mathbb{Z}$ implies $c(n,r) = c(n + rN\lambda + mN^2\lambda^2, r + 2mN\lambda)$ for all $\lambda \in \mathbb{Z}$.

Let p be a prime. Similarly as in [5], we define the p -th Hecke operator on $J_{k,m}(\Gamma_0(N^2) \ltimes (N\mathbb{Z} \times \mathbb{Z}), \chi)$ by

$$\phi|T_p = p^{k-4} \sum_{\substack{M \in \Gamma_0(N^2) \setminus M_0(N^2) \\ \det(M)=p^2 \\ \gcd(M)=\square}} \chi(a) \sum_{\lambda, \mu(p)} \phi|_{k,m} \left[\frac{1}{p}M, [N\lambda, \mu], 1 \right],$$

where $M_0(N^2)$ is the set of integral 2×2 matrices whose lower left entry is divisible by N^2 , and $\gcd(M) = \square$ means that the greatest common divisor of the entries of M is a square.

Lemma 2.2. The Hecke operator T_p defines an endomorphism of $J_{k,m}(\Gamma_0(N^2) \ltimes (N\mathbb{Z} \times \mathbb{Z}), \chi)$ which maps cusp forms to cusp forms. The (n,r) -th coefficient of $\phi|T_p = \sum_{r^2 \leq 4mn} c^*(n,r) q^n \zeta^r$ is given by

$$\begin{aligned} c^*(n,r) &= c(p^2n, pr) + \chi(p)C_p(n,r,m)p^{k-3}c(n,r) \\ &\quad + \chi(p^2)p^{2k-3} \sum_{\lambda(p)} c((n + rN\lambda + mN^2\lambda^2)/p^2, (r + 2mN\lambda)/p), \end{aligned}$$

where

$$C_p(n,r,m) = \begin{cases} p \left(\frac{D}{p} \right), & \text{if } p \nmid m, \\ 0, & \text{if } p \mid m; \quad p \nmid r, \\ -p, & \text{if } p \mid m, r; \quad p \nmid n, \\ p(p-1), & \text{if } p \mid m, r, n. \end{cases}$$

Here we use the convention $c(n,r) = 0$ if n or r is not integral.

Proof. The well-definedness and the mapping property of T_p are easy exercises, so we only compute the Fourier expansion of $\phi|T_p$. As representatives for $\Gamma_0(N^2) \setminus M_0(N^2)$ with determinant p^2 and square content we use the matrices

$$\alpha_b = \begin{pmatrix} 1 & b \\ 0 & p^2 \end{pmatrix}, \quad 0 \leq b < p^2, \quad \beta_h = \begin{pmatrix} p & h \\ 0 & p \end{pmatrix}, \quad 0 < h < p, \quad \sigma = \begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix},$$

compare Theorem 1.7 in [13]. We compute the contribution to $c^*(n, r)$ for each of the three types of representatives individually.

We first decompose $\left[\frac{1}{p}\alpha_b, [N\lambda, \mu], 1\right]$ as

$$\left[\begin{pmatrix} 1/p & b/p \\ 0 & p \end{pmatrix}, [N\lambda, \mu], 1\right] = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, [N\lambda p, 0], 1\right] \cdot \left[\begin{pmatrix} 1/p & b/p \\ 0 & p \end{pmatrix}, [0, \mu - N\lambda b], 1\right].$$

Since ϕ is invariant under the slash operation of the first element on the right-hand side, the α_b -part of $\phi|T_p$ is given by

$$\begin{aligned} & p^{k-4} \sum_{b(p^2)} \sum_{\lambda, \mu(p)} \phi \Big|_{k, m} \left[\begin{pmatrix} 1/p & b/p \\ 0 & p \end{pmatrix}, [N\lambda p, 0], 1\right] \\ &= p^{k-4} \sum_{b(p^2)} \sum_{\lambda, \mu(p)} \phi \Big|_{k, m} \left[\begin{pmatrix} 1/p & b/p \\ 0 & p \end{pmatrix}, [0, \mu - N\lambda b], 1\right] \\ &= p^{k-4} \sum_{b(p^2)} \sum_{\lambda, \mu(p)} p^{-k} \phi((\tau + b)/p^2, (z + \mu - N\lambda b)/p). \end{aligned}$$

If we replace $\mu - N\lambda b$ by μ and plug in the Fourier expansion of ϕ , the last line becomes

$$p^{-3} \sum_{b(p^2)} \sum_{\mu(p)} \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 \leq 4mn}} c(n, r) e(nb/p^2) e(r\mu/p) q^{n/p^2} \zeta^{r/p}.$$

Now the product of the two exponential sums over b and μ gives a factor p^3 if n is divisible by p^2 and r is divisible by p , and vanishes otherwise. Thus we get the contribution

$$\sum_{\substack{n, r \in \mathbb{Z} \\ r^2 \leq 4mn \\ p^2 | n, p | r}} c(n, r) q^{n/p^2} \zeta^{r/p} = \sum_{\substack{n', r' \in \mathbb{Z} \\ r'^2 \leq 4mn'}} c(p^2 n', pr') q^{n'} \zeta^{r'}$$

of the α_b -part, where we replaced $n = p^2 n'$ and $r = pr'$.

For $\left[\frac{1}{p}\beta_h, [N\lambda, \mu], 1\right]$ we have

$$\left[\begin{pmatrix} 1 & h/p \\ 0 & 1 \end{pmatrix}, [N\lambda, \mu], 1\right] = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, [N\lambda, \mu], 1\right] \cdot \left[\begin{pmatrix} 1 & h/p \\ 0 & 1 \end{pmatrix}, [0, -N\lambda h/p], e(N^2 \lambda^2 h/p)\right].$$

Using the invariance of ϕ under the slash operation of the first element on the right-hand side we see that the β_h -part of $\phi|T_p$ equals $\chi(p)$ times

$$\begin{aligned} & p^{k-4} \sum_{h(p)^*} \sum_{\lambda, \mu(p)} \phi \Big|_{k,m} \left[\begin{pmatrix} 1 & h/p \\ 0 & 1 \end{pmatrix}, [N\lambda, \mu], 1 \right] \\ &= p^{k-3} \sum_{h(p)^*} \sum_{\lambda(p)} \phi \Big|_{k,m} \left[\begin{pmatrix} 1 & h/p \\ 0 & 1 \end{pmatrix}, [0, -N\lambda h/p], e(N^2\lambda^2 h/p) \right] \\ &= p^{k-3} \sum_{h(p)^*} \sum_{\lambda(p)} e^m(N^2\lambda^2 h/p) \phi(\tau + h/p, z - N\lambda h/p). \end{aligned}$$

Plugging in the Fourier expansion of ϕ we get

$$p^{k-3} \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 \leq 4mn}} c(n, r) \left(\sum_{h(p)^*} \sum_{\lambda(p)} e(h(n - rN\lambda + mN^2\lambda^2)/p) \right) q^n \zeta^r.$$

The determination of the inner double sum is a standard computation involving quadratic Gauss sums, so we omit it for brevity. It equals the constant $C_p(n, r, m)$ defined above.

The σ -part of $\phi|T_p$ is $\chi(p^2)$ times

$$\begin{aligned} & p^{k-4} \sum_{\lambda, \mu(p)} \phi \Big|_{k,m} \left[\begin{pmatrix} p & 0 \\ 0 & 1/p \end{pmatrix}, [N\lambda, \mu], 1 \right] \\ &= p^{k-4} \sum_{\lambda, \mu(p)} p^k e^m(N^2\lambda^2 \tau + 2N\lambda z + N\lambda \mu) \phi(p^2 \tau, p(z + N\lambda \tau + \mu)). \end{aligned}$$

Since ϕ is periodic in the second argument and λ, μ are integers, the sum over μ gives a factor p . Inserting the Fourier expansion we get

$$= p^{2k-3} \sum_{\lambda(p)} \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 \leq 4mn}} c(n, r) q^{np^2 + rpN\lambda + mN^2\lambda^2} \zeta^{rp + 2mN\lambda}.$$

We set $n' = np^2 + rpN\lambda + mN^2\lambda^2$ and $r' = rp + 2mN\lambda$. Then we find $r = (r' - 2mN\lambda)/p$ and, after a short calculation, $n = (n' - r'N\lambda + mN^2\lambda^2)/p^2$. Finally, replacing λ by $-\lambda$, the last line becomes

$$= p^{2k-3} \sum_{\substack{n', r' \in \mathbb{Z} \\ r'^2 \leq 4mn'}} \sum_{\lambda(p)} c((n' + r'N\lambda + mN^2\lambda^2)/p^2, (r' + 2mN\lambda)/p) q^{n'} \zeta^{r'}.$$

This completes the proof. \square

Next, we define a Fricke involution on $J_{k,m}(\Gamma_0(N^2) \ltimes (N\mathbb{Z} \times \mathbb{Z}), \chi)$.

Lemma 2.3. *Let $\phi \in J_{k,m}(\Gamma_0(N^2) \ltimes (N\mathbb{Z} \times \mathbb{Z}), \chi)$. Then*

$$\phi|W_{N^2} := \phi \Big|_{k,m} \begin{pmatrix} 0 & -1/N \\ N & 0 \end{pmatrix}$$

is in $J_{k,m}(\Gamma_0(N^2) \ltimes (N\mathbb{Z} \times \mathbb{Z}), \overline{\chi})$ and it holds $\phi|W_{N^2}|W_{N^2} = \chi(-1)\phi$. If ϕ is a cusp form, so is $\phi|W_{N^2}$.

Proof. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N^2)$ and $[\lambda, \mu] \in N\mathbb{Z} \times \mathbb{Z}$ we have

$$W_{N^2} \cdot \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, [\lambda, \mu], 1 \right] = \left[\begin{pmatrix} d & -c/N^2 \\ -bN^2 & a \end{pmatrix}, [-\mu N, \lambda/N], 1 \right] \cdot W_{N^2}.$$

The transformation behaviour now follows from $\chi(a) = \overline{\chi}(d)$ as $ad \equiv 1 \pmod{N^2}$. The remaining properties are obvious. \square

The existence of this nice Fricke involution is the main reason why we work with the group $\Gamma_0(N^2) \ltimes (N\mathbb{Z} \times \mathbb{Z})$. It is easy to check that for $(p, 2mN) = 1$ we have the commutation relation

$$\phi|W_{N^2}|T_p = \overline{\chi}(p^2)\phi|T_p|W_{N^2}.$$

We now introduce a twisting and a projection operator.

Lemma 2.4. *Let $\phi = \sum_{r^2 \leq 4mn} c(n, r)q^n \zeta^r \in J_{k,m}(\Gamma_0(N^2) \ltimes (N\mathbb{Z} \times \mathbb{Z}), \chi)$. Further, let p be a positive integer.*

1 *Let ψ be a primitive Dirichlet character mod p . Then the twisted function*

$$\phi_\psi(\tau, z) := \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 \leq 4mn}} \psi(r^2 - 4mn) c(n, r) q^n \zeta^r$$

is in $J_{k,m}(\Gamma_0(N^2 p^2) \ltimes (Np\mathbb{Z} \times \mathbb{Z}), \chi \psi^2)$.

2 *The projected function*

$$\phi|B_p(\tau, z) := \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 \leq 4mn \\ p|(r^2 - 4mn)}} c(n, r) q^n \zeta^r$$

is in $J_{k,m}(\Gamma_0(N^2 p^2) \ltimes (Np\mathbb{Z} \times \mathbb{Z}), \chi)$.

Both operators map cusp forms to cusp forms.

Proof. First note that the Fourier coefficients of ϕ_ψ and $\phi|B_p$ are invariant under $n \mapsto n + rNp\lambda + mN^2 p^2 \lambda^2, r \mapsto r + 2mNp\lambda$ for $\lambda \in \mathbb{Z}$, which is equivalent to the invariance of ϕ under $[\lambda, \mu] \in Np\mathbb{Z} \times \mathbb{Z}$.

We check the transformation behaviour of ϕ_ψ under $\Gamma_0(N^2 p^2)$. Since ψ is primitive, we can write

$$\psi(n) = \frac{1}{G(\overline{\psi})} \sum_{\alpha(p)^*} \overline{\psi}(\alpha) e(n\alpha/p)$$

with the Gauss sum $G(\bar{\psi}) = \sum_{\alpha(p)^*} \bar{\psi}(\alpha) e(\alpha/p)$. We compute

$$\begin{aligned} \phi_{\psi}(\tau, z) &= \frac{1}{G(\bar{\psi})} \sum_{\alpha(p)^*} \bar{\psi}(\alpha) \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 \leq 4mn}} c(n, r) e((r^2 - 4mn)\alpha/p) q^n \zeta^r \\ &= \frac{1}{G(\bar{\psi})} \sum_{\alpha(p)^*} \bar{\psi}(\alpha) \sum_{\substack{\rho(p) \\ n, r \in \mathbb{Z} \\ r^2 \leq 4mn \\ r \equiv \rho(p)}} c(n, r) e((r^2 - 4mn)\alpha/p) q^n \zeta^r \\ &= \frac{1}{G(\bar{\psi})} \sum_{\alpha(p)^*} \bar{\psi}(\alpha) \sum_{\rho(p)} e(\rho^2 \alpha/p) \\ &\quad \times \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 \leq 4mn}} c(n, r) \left(\frac{1}{p} \sum_{\beta(p)} e((\rho - r)\beta/p) \right) e(-4mn\alpha/p) q^n \zeta^r \\ &= \frac{1}{pG(\bar{\psi})} \sum_{\alpha(p)^*} \bar{\psi}(\alpha) \sum_{\beta(p)} G(\alpha, \beta, p) \\ &\quad \times \phi(\tau, z) \Big|_{m, k} \left[\begin{pmatrix} 1 & -4m\alpha/p \\ 0 & 1 \end{pmatrix}, [0, -\beta/p], 1 \right], \end{aligned}$$

where $G(\alpha, \beta, p) = \sum_{\rho(p)} e((\rho^2 \alpha + \rho \beta)/p)$ is a quadratic Gauss sum. For $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N^2 p^2)$ we have

$$\begin{aligned} &\left[\begin{pmatrix} 1 & -4m\alpha/p \\ 0 & 1 \end{pmatrix}, [0, -\beta/p], 1 \right] \cdot [M, [0, 0], 1] \\ &= [M', [\lambda', \mu'], 1] \cdot \left[\begin{pmatrix} 1 & -4md^2\alpha/p \\ 0 & 1 \end{pmatrix}, [0, -d\beta/p], 1 \right] \end{aligned}$$

where

$$M' = \begin{pmatrix} a - 4mc\alpha/p & b + 4md\alpha(ad - 1)/p - 16m^2cd^2\alpha^2/p^2 \\ c & d + 4mc\alpha d^2/p \end{pmatrix} \in \Gamma_0(N^2 p^2)$$

and

$$[\lambda', \mu'] = [-c\beta/p, 0] \begin{pmatrix} 1 & 4md^2\alpha/p \\ 0 & 1 \end{pmatrix} \in Np\mathbb{Z} \times \mathbb{Z}.$$

Note that $\chi(M') = \chi(M)$ since χ is a character mod N^2 and c/p is divisible by N^2 . Let $\bar{d} \in \mathbb{Z}$ with $d\bar{d} \equiv 1 \pmod{p}$. Replacing α by $\bar{d}^2\alpha$ and β by $\bar{d}\beta$, and using the elementary property $G(\bar{d}^2\alpha, \bar{d}\beta, p) = G(\alpha, \beta, p)$ of the Gauss sum, we find

$$\begin{aligned} &\phi_{\psi}|_{k, m} M \\ &= \frac{1}{pG(\bar{\psi})} \chi(M) \sum_{\alpha(p)^*} \bar{\psi}(\alpha) \sum_{\beta(p)} G(\alpha, \beta, p) \phi(\tau - 4md^2\alpha/p, z - d\beta/p) \\ &= \frac{1}{pG(\bar{\psi})} \chi(M) \sum_{\alpha(p)^*} \bar{\psi}(\bar{d}^2\alpha) \sum_{\beta(p)} G(\bar{d}^2\alpha, \bar{d}\beta, p) \phi(\tau - 4m\alpha/p, z - \beta/p) \\ &= \chi(M) \psi(d)^2 \frac{1}{pG(\bar{\psi})} \sum_{\alpha(p)^*} \bar{\psi}(\alpha) \sum_{\beta(p)} G(\alpha, \beta, p) \phi(\tau - 4m\alpha/p, z - \beta/p) \\ &= \chi(M) \psi(M)^2 \phi_{\psi}. \end{aligned}$$

This shows that ϕ_ψ transforms correctly. A standard argument shows that ϕ_ψ has suitable Fourier expansions at the cusps.

For $\phi|B_p$ a similiar calculation as above gives

$$\phi|B_p(\tau, z) = \frac{1}{p^2} \sum_{\alpha(p)} \sum_{\beta(p)} G(\alpha, \beta, p) \phi(\tau, z) \Big|_{m,k} \left[\begin{pmatrix} 1 & -4m\alpha/p \\ 0 & 1 \end{pmatrix}, [0, -\beta/p], 1 \right].$$

Now we proceed in the same way as before to get the transformation law for $\phi|B_p$ under $\Gamma_0(N^2 p^2)$. \square

One could treat twists of Jacobi forms by $\chi(n)$ or $\chi(r)$ in a similar way. The latter, for example, has been considered in [10].

Let $\phi \in J_{k,m}(\Gamma_0(N^2) \ltimes (N\mathbb{Z} \times \mathbb{Z}), \chi)$ and let ψ be a quadratic character mod p . For each prime q with $(q, p) = 1$ it holds

$$\begin{aligned} (\phi|T_q)_\psi &= \phi_\psi|T_q, \\ \phi|T_q|B_p &= \phi|B_p|T_q. \end{aligned}$$

This can be checked by comparing the actions of the operators on Fourier expansions. The interplay between the Fricke involution and the twisting and projection operators is described by the following lemma.

Lemma 2.5. *Let p be a prime with $(p, 2mN) = 1$. For $\phi \in J_{k,m}(\Gamma_0(N^2) \ltimes (N\mathbb{Z} \times \mathbb{Z}), \chi)$ and $\psi = \left(\frac{\cdot}{p}\right)$ we have*

$$(\phi|W_{N^2}^{-1})_\psi|W_{N^2 p^2} = \overline{\chi}(p)p \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 \leq 4mn \\ p|n,r}} c(n,r)q^n \zeta^r - \overline{\chi}(p) \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 \leq 4mn \\ p|r}} c(n,r)q^n \zeta^r$$

and

$$\phi|W_{N^2}^{-1}|B_p|W_{N^2 p^2} = p^{k-1} \phi(p^2 \tau, pz) + \overline{\chi}(p) \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 \leq 4mn \\ p|r}} \left(\frac{r^2 - 4mn}{p} \right) c(n,r)q^n \zeta^r.$$

Proof. Using the representation of ϕ_ψ obtained in the proof of Lemma 2.4 we have

$$\begin{aligned} (\phi|W_{N^2}^{-1})_\psi|W_{N^2 p^2} &= \frac{\psi(-m)}{pG(\overline{\psi})} \sum_{\alpha(p)^*} \overline{\psi}(\alpha) \sum_{\beta(p)} \left(\sum_{\rho(p)} e(-m(\rho^2 \alpha + 2\rho\beta)/p) \right) \\ &\quad \times \phi \Big|_{k,m} W_{N^2}^{-1} \left[\begin{pmatrix} 1 & \alpha/p \\ 0 & 1 \end{pmatrix}, [0, \beta/p], 1 \right] W_{N^2 p^2}, \end{aligned}$$

where we replaced $-4m\alpha$ by α , $-\beta$ by β and ρ by $2m\rho \bmod p$. Let $\overline{N^2\alpha} \in \mathbb{Z}$ denote an inverse of $N^2\alpha \bmod p$. We have

$$\begin{aligned} & W_{N^2}^{-1} \left[\begin{pmatrix} 1 & \alpha/p \\ 0 & 1 \end{pmatrix}, [0, \beta/p], 1 \right] W_{N^2 p^2} \\ &= \left[\begin{pmatrix} p & \overline{N^2\alpha} \\ -N^2\alpha & (1 - \overline{N^2\alpha} N^2\alpha)/p \end{pmatrix}, [N\beta, 0], 1 \right] \\ &\quad \cdot \left[\begin{pmatrix} 1 & -\overline{N^2\alpha}/p \\ 0 & 1 \end{pmatrix}, [0, \overline{N^2\alpha} N\beta/p], e(-\overline{\alpha}\beta^2/p) \right]. \end{aligned}$$

We obtain

$$\begin{aligned} (\phi|W_{N^2}^{-1})_\psi|W_{N^2 p^2} &= \frac{\overline{\chi}(p)}{pG(\overline{\psi})} \psi(-m) \sum_{\alpha(p)^*} \overline{\psi}(\alpha) \\ &\quad \times \sum_{\beta(p)} \left(\sum_{\rho(p)} e(-m(\rho^2\alpha + 2\rho\beta + \overline{\alpha}\beta^2)/p) \right) \\ &\quad \times \phi \Big|_{k,m} \left[\begin{pmatrix} 1 & -\overline{N^2\alpha}/p \\ 0 & 1 \end{pmatrix}, [0, \overline{N^2\alpha}\beta/p], 1 \right]. \end{aligned}$$

The sum over ρ is just $\psi(-m\alpha)G(\psi)$. Since $\psi = \overline{\psi}$, we find

$$(\phi|W_{N^2}^{-1})_\psi|W_{N^2 p^2} = \frac{\overline{\chi}(p)}{p} \sum_{\alpha(p)^*} \sum_{\beta(p)} \phi \Big|_{k,m} \left[\begin{pmatrix} 1 & \alpha/p \\ 0 & 1 \end{pmatrix}, [0, \beta/p], 1 \right].$$

Plugging in the Fourier expansion of ϕ and an easy calculation give the stated formula.

For the projection operator we use the representation for $\phi|B_p$ given at the end of the proof of Lemma 2.4. The summand for $\alpha = 0$ gives $p^{k-1}\phi(p^2\tau, pz)$ and the remaining part can be treated as above. \square

3 Proofs of the main results

We start with the proof of Theorem 1.2 as it will be needed in the proof of Theorem 1.1. Recall that $v_\ell(\phi) = \inf_{n,r} v_\ell(c(n, r))$.

Proof of Theorem 1.2. First note that $\phi|U_N(\tau, z) = \phi(\tau, Nz)$ is in $J_{k,mN^2}(\Gamma_1(N^2) \ltimes \mathbb{Z}^2)$ and $\phi|U_N|_{k,mN^2}M = \phi|_{k,m}M|U_N$. Since $\phi|_{k,m}M$ and $\phi|_{k,m}M|U_N$ have the same set of Fourier coefficients, it suffices to show the claim for $\phi \in J_{k,mN^2}(\Gamma_1(N^2) \ltimes \mathbb{Z}^2)$. To simplify the notation, we take $\phi \in J_{k,m}(\Gamma_1(N) \ltimes \mathbb{Z}^2)$.

Such a form ϕ has a theta decomposition

$$\phi(\tau, z) = \sum_{\mu(2m)} h_\mu(\tau) \theta_{m,\mu}(\tau, z)$$

with

$$h_\mu(\tau) = \sum_{D < 0} c_\mu(D) q^{-D/4m}, \quad \theta_{m,\mu}(\tau, z) = \sum_{\substack{r \in \mathbb{Z} \\ r \equiv \mu(2m)}} q^{r^2/4m} \zeta^r,$$

where $c_\mu(D) = c((\mu^2 - D)/4m, \mu)$, see [5], §5. The theta function $\theta_{m,\mu}$ satisfies the transformation rules

$$\begin{aligned}\theta_{m,\mu}(\tau + 1, z) &= e(\mu^2/4m)\theta_{m,\mu}, \\ \theta_{m,\mu}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) &= \sqrt{\tau/2mi} e(mz^2/\tau) \sum_{v(2m)} e(-\mu v/2m)\theta_{m,v}.\end{aligned}$$

This together with the transformation behaviour of ϕ under $\Gamma_1(N)$ implies that the tuple $(h_\mu)_{\mu(2m)}$ transforms under $\Gamma_1(N)$ as a vector valued modular form of weight $k - 1/2$ for the Weil representation ρ_L of the lattice $L = \mathbb{Z}$ with quadratic form $q(x) = -mx^2$, compare [2], Section 1.1. For $M \in SL_2(\mathbb{Z})$ we have

$$(\phi|_{k,m}M)(\tau, z) = \sum_{\mu(2m)} (h_\mu|_{k-1/2}M)(\tau) \cdot (\theta_{m,\mu}|_{1/2,m}M)(\tau, z).$$

From the transformation rules of $\theta_{m,\mu}$ stated above it follows that $\theta_{m,\mu}|_{1/2,m}M = \sum_{v(2m)} \rho_{v,\mu} \theta_{m,v}$ with some algebraic numbers $\rho_{v,\mu}$ with non-negative ℓ -adic valuation. Here the assumption $(\ell, 2mN) = 1$ was used. Thus to prove the claim of the theorem it remains to show that the functions $h_\mu|_{k-1/2}M$ have algebraic coefficients with $v_\ell(h_\mu|_{k-1/2}M) \geq v_\ell(h_\mu)$.

By the explicit formula for the action of $\Gamma_1(4m)$ in the Weil representation ρ_L given in [12], Proposition 4.5., the functions h_μ are modular forms of weight $k - 1/2$ for $\Gamma_1(4mN)$ and character $\chi_\mu \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = e(-b\mu^2/4m)$. We write

$$h_\mu|_{k-1/2}M = \left(h_\mu \Big|_{k-1/2} \begin{pmatrix} 2m & 0 \\ 0 & \frac{1}{2m} \end{pmatrix} \right) \Big|_{k-1/2} \begin{pmatrix} \frac{1}{2m} & 0 \\ 0 & 2m \end{pmatrix} M.$$

The inner function is now a modular form of weight $k - 1/2$ for $\Gamma_1(16m^3N)$ and trivial character with algebraic Fourier coefficients and the same ℓ -adic valuation as h_μ . The product of the remaining two matrices can be written as $M' \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$ with $M' \in SL_2(\mathbb{Z})$ and $\alpha, \beta, \delta \in \mathbb{Q}$ with $v_\ell(\delta) = 0$. It now follows from the q -expansion principle (see [7], Corollary 1.6.2., or [4], Remark 12.3.5., for the q -expansion principle for integral weight modular forms, and the proof of Lemma 1 in [1] for the transition to half-integral weight forms) that $h_\mu|_{k-1/2}M$ has algebraic coefficients with $v_\ell(h_\mu|_{k-1/2}M) \geq v_\ell(h_\mu)$. The assumption $(\ell, 2mN) = 1$ is again needed here. \square

Proposition 3.1. *Let p be a prime with $(p, 2mN) = 1$ and let $\psi = \left(\frac{\cdot}{p} \right)$. Suppose that $\phi \in J_{k,m}(\Gamma_0(N^2) \times (N\mathbb{Z} \times \mathbb{Z}), \chi)$ has integral algebraic Fourier coefficients. Let ℓ be a prime with $(\ell, 2mNp(p-1)) = 1$, let $\varepsilon \in \{\pm 1\}$ and let*

$$h := \phi|(1 - B_p) - \varepsilon\phi_\psi = 2 \sum_{\substack{n,r \in \mathbb{Z} \\ \psi(r^2 - 4mn) = -\varepsilon}} c(n, r) q^n \zeta^r.$$

Then $v_\ell(\phi|T_p - \chi(p)\varepsilon(p^{k-1} + p^{k-2})\phi) \geq v_\ell(h)$. In particular, if ϕ is an eigenform of T_p with eigenvalue λ_p and $v_\ell(\phi) = 0$, there is a discriminant $D = r^2 - 4mn < 0$ such that $\left(\frac{D}{p} \right) = -\varepsilon$ and $v_\ell(c(n, r)) \leq v_\ell(\lambda_p - \chi(p)\varepsilon(p^{k-1} + p^{k-2}))$.

Proof. Let $h' = h|W_{N^2p^2}$. If we write $\begin{pmatrix} 0 & -1/Np \\ Np & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} Np & 0 \\ 0 & 1/Np \end{pmatrix}$, Theorem 1.2 implies that h' has algebraic coefficients with $v := v_\ell(h) = v_\ell(h')$. We get equality instead of an inequality here since $W_{N^2p^2}$ is an involution (up to a sign). Further, let $c'(n, r)$ denote the Fourier coefficients of $\phi' = \phi|W_{N^2}$, which are again algebraic. We write

$$h' = \phi'|W_{N^2}^{-1}|W_{N^2p^2} - \phi'|W_{N^2}^{-1}|B_p|W_{N^2p^2} - \varepsilon(\phi'|W_{N^2}^{-1})_\psi|W_{N^2p^2}.$$

It is easily checked that $\phi'|W_{N^2}^{-1}|W_{N^2p^2}(\tau, z) = p^k \phi'(p^2\tau, pz)$. Since ϕ' transforms with character $\bar{\chi}$, we get from Lemma 2.5 the formula

$$\begin{aligned} h' &= (p^k - p^{k-1}) \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 \leq 4mn}} c'(n/p^2, r/p) q^n \zeta^r - \chi(p) \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 \leq 4mn \\ p|r}} \left(\frac{D}{p}\right) c'(n, r) q^n \zeta^r \\ &\quad - \chi(p) \varepsilon p \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 \leq 4mn \\ p|n, r}} c'(n, r) q^n \zeta^r + \chi(p) \varepsilon \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 \leq 4mn \\ p|r}} c'(n, r) q^n \zeta^r, \end{aligned}$$

where $D = r^2 - 4mn$. This gives the relations

$$v_\ell \left(\left(\frac{D}{p}\right) c'(n, r) - \varepsilon c'(n, r) \right) \geq v, \quad \text{if } p \mid r, p \nmid n, \quad (2)$$

$$v_\ell(c'(n, r)) \geq v, \quad \text{if } p \mid r, p \mid n, p^2 \nmid n, \quad (3)$$

$$v_\ell(c'(n, r) - \bar{\chi}(p) \varepsilon p^{k-1} c'(n/p^2, r/p)) \geq v, \quad \text{if } p \mid r, p^2 \mid n, \quad (4)$$

where we divided by $\chi(p)$, ε , p^{k-1} or $p-1$ at some places. Using $c'(n, r) = c'(n + rN\lambda + mN^2\lambda^2, r + 2mN\lambda)$ for $\lambda \in \mathbb{Z}$ and the assumption $(p, 2mN) = 1$ we may assume that $p \mid r$. For $p \mid r$ and $(p, 2mN) = 1$, the coefficient $c'^*(n, r)$ of $\phi'|T_p$ is given by

$$c'^*(n, r) = c'(p^2n, pr) + \bar{\chi}(p) \left(\frac{D}{p}\right) p^{k-2} c'(n, r) + \bar{\chi}(p^2) p^{2k-3} c'(n/p^2, r/p).$$

Using the relations (2), (3), (4) it is straightforward to verify that

$$v_\ell(c'^*(n, r) - \bar{\chi}(p) \varepsilon (p^{k-1} + p^{k-2}) c'(n, r)) \geq v.$$

The result now follows from $\phi'|T_p = \phi|W_{N^2}^{-1}|T_p = \bar{\chi}(p^2) \phi|T_p|W_{N^2}^{-1}$ and Theorem 1.2. \square

Proposition 3.2. *Let p be a prime with $(p, 2mN) = 1$ and let $\phi \in J_{k,m}(\Gamma_0(N^2) \ltimes (N\mathbb{Z} \times \mathbb{Z}), \chi)$ with integral algebraic Fourier coefficients which is an eigenform of T_p with eigenvalue λ_p . Let ℓ be a prime with $(\ell, 2mNp(p-1)) = 1$ and let $\varepsilon \in \{\pm 1\}$. Suppose that there is some discriminant $D^* = r^{*2} - 4mn^* < 0$ with $\left(\frac{D^*}{p}\right) = \varepsilon$ and $v_\ell(c(n^*, r^*)) = 0$. Then there exist infinitely many discriminants $D = r^2 - 4mn < 0$ with pairwise distinct square free parts and $\left(\frac{D}{p}\right) = -\varepsilon$ such that $v_\ell(c(n, r)) \leq v_\ell(\lambda_p - \chi(p) \varepsilon (p^{k-1} + p^{k-2}))$.*

Proof. We prove by induction on t that there are negative discriminants $D_1 = r_1^2 - 4mn_1, \dots, D_t = r_t^2 - 4mn_t$ with pairwise distinct square free parts such that $\left(\frac{D_j}{p}\right) = -\varepsilon$ and $v_\ell(c(n_j, r_j)) \leq v_\ell(\lambda_p - \chi(p) \varepsilon (p^{k-1} + p^{k-2}))$ for $j = 1, \dots, t$.

For $t = 1$ the statement follows from the last proposition. Now let us assume that for ϕ as above and fixed $t > 1$ we can always find $t - 1$ discriminants D_2, \dots, D_t with the

desired properties. By the last proposition we find a discriminant $D_1 = r_1^2 - 4mn_1 < 0$ with $\left(\frac{D_1}{p}\right) = -\varepsilon$ and $v_\ell(c(n_1, r_1)) \leq v_\ell(\lambda_p - \chi(p)\varepsilon(p^{k-1} + p^{k-2}))$. Since $\left(\frac{D_1}{p}\right) = -\varepsilon$ and $\left(\frac{D^*}{p}\right) = \varepsilon$ the square free parts of D_1 and D^* are different. Thus we can choose a prime q with

$$(q, \ell) = 1, \quad (q, p) = 1, \quad \left(\frac{D^*}{q}\right) = +\varepsilon, \quad \left(\frac{D_1}{q}\right) = -\varepsilon.$$

Put $\psi = \left(\frac{\cdot}{q}\right)$ and consider the function

$$\phi_1 = \frac{1}{2}(\phi|(1 - B_q) + \varepsilon\phi_\psi) \in J_{k,m}(\Gamma_0(N^2q^2) \times (Nq\mathbb{Z} \times \mathbb{Z}), \chi\psi^2).$$

Then ϕ_1 has the Fourier expansion

$$\phi_1 = \sum_{\substack{n,r \in \mathbb{Z} \\ r^2 \leq 4mn}} c_1(n, r) q^n \zeta^r, \quad \text{with } c_1(n, r) = \begin{cases} c(n, r), & \text{if } \left(\frac{r^2 - 4mn}{q}\right) = \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

Since T_p commutes with the twisting and projection operators in the definition of ϕ_1 , the function ϕ_1 is also an eigenform of T_p with eigenvalue λ_p . Further, $c_1(n^*, r^*) = c(n^*, r^*)$, so $v_\ell(\phi_1) = v_\ell(\phi) = 0$. We can apply the induction hypothesis on ϕ_1 to find discriminants $D_2 = r_2^2 - 4mn_2, \dots, D_t = r_t^2 - 4mn_t$ with pairwise distinct square free parts such that $\left(\frac{D_i}{p}\right) = -\varepsilon$ and $v_\ell(c(n_j, r_j)) = v_\ell(c_1(n_j, r_j)) \leq v_\ell(\lambda_p - \chi(p)\varepsilon(p^{k-1} + p^{k-2}))$. Since $\left(\frac{D_1}{q}\right) = -\varepsilon$, we have $c_1(n, r) = 0$ for all n, r such that $r^2 - 4mn$ has the same square free part as D_1 , so the square free parts of D_2, \dots, D_t are different from the square free part of D_1 . This completes the proof. \square

So far, the discriminants given in Proposition 3.2 need not be fundamental. To find fundamental discriminants with the properties as in the proposition, we will make use of the relations given by the Hecke eigenform equations.

Lemma 3.3. *Let $\phi \in J_{k,m}(\Gamma_0(N^2) \times (N\mathbb{Z} \times \mathbb{Z}), \chi)$ be an eigenform of all Hecke operators T_p , $(p, mN) = 1$, with algebraic Fourier coefficients $c(n, r)$. Let ℓ be a prime and $D = r^2 - 4mn \leq 0$ a fundamental discriminant. Then $v_\ell(c(f^2n, fr)) \geq v_\ell(c(n, r))$ for all $f \in \mathbb{Z}$ with $(f, mN) = 1$.*

Proof. By induction on $|f|$: For $f = 1$ there is nothing to show and for $f = -1$ we have $c(n, r) = (-1)^k c(n, -r)$. For $|f| > 1$ we choose a prime divisor p of f and write $f = pf_1$. The action of T_p on ϕ gives the relation

$$\begin{aligned} & \lambda_p c(f_1^2 n, f_1 r) \\ &= c(f^2 n, fr) + \chi(p) \left(\frac{f_1^2 D}{p}\right) p^{k-2} c(f_1^2 n, f_1 r) \\ & \quad + \chi(p^2) p^{2k-3} \sum_{\lambda(p)} c((f_1^2 n + f_1 r N \lambda + m N^2 \lambda^2)/p^2, (f_1 r + 2m N \lambda)/p). \end{aligned}$$

We find

$$\begin{aligned} & v_\ell(c(f^2n, fr)) \\ & \geq \min \left(v_\ell(\lambda_q c(f_1^2n, f_1r)), v_\ell(p^{k-2}c(f_1^2n, f_1r)), \right. \\ & \quad \left. v_\ell \left(p^{2k-3} \sum_{\lambda(p)} c((f_1^2n + f_1rN\lambda + mN^2\lambda^2)/p^2, (f_1r + 2mN\lambda)/p) \right) \right). \end{aligned}$$

In the sum over λ the summands can be nonzero only if $p \mid f_1$ since D is fundamental, in which case the sum is just $c((f_1/p)^2n, (f_1/p)r)$ as $(p, mN) = 1$. Note that λ_p is an algebraic integer, so all the terms in the minimum are at least $v_\ell(c(n, r))$ by the induction hypothesis, which proves the claim. \square

We would now like to write each $D = r^2 - 4mn$ in Proposition 3.2 as $D = f^2D_0$ with a fundamental discriminant $D_0 = r_0^2 - 4mn_0$ and $f \in \mathbb{Z}$, and then use the relations given by the eigenform equations to show that $v_\ell(c(n_0, r_0)) \leq v_\ell(c(n, r))$. But in general, D_0 will not necessarily be a square mod $4m$, as the example $m = 3$ and $D = 0^2 - 4 \cdot 3 \cdot 3 = 9 \cdot (-4)$ shows, and even if D_0 is a square mod $4m$, the eigenform equations will in general not give information about the relation of $c(n, r)$ and $c(f^2n_0, fr_0)$ if $r \not\equiv fr_0 \pmod{2m}$. But these problems can only arise if $(D, mN) > 1$, as the following lemma shows.

Lemma 3.4. *Let $\phi \in J_{k,m}(\Gamma_0(N^2) \times (N\mathbb{Z} \times \mathbb{Z}), \chi)$ with Fourier coefficients $c(n, r)$. Let $D = r^2 - 4mn < 0$ be a discriminant with $(D, mN) = 1$ and write $D = f^2D_0$ with a fundamental discriminant $D_0 < 0$ and some $f \in \mathbb{Z}$. Then D_0 is a square mod $4m$, i.e. $D_0 = r_0^2 - 4mn_0$ for some $r_0, n_0 \in \mathbb{Z}$, and $c(n, r) = c(f^2n_0, fr_0)$.*

Proof. It suffices to show the claim in the case that $f = p$ is a prime with $(p, mN) = 1$ and $D_0 < 0$ an arbitrary (i.e. not necessarily fundamental) discriminant. We will use that $c(n, r) = c(n + rN\lambda + mN^2\lambda^2, r + 2mN\lambda)$ holds for every $\lambda \in \mathbb{Z}$.

Assume that p is odd. Since $2mN$ is invertible mod p we can achieve $r + 2mN\lambda \equiv 0(p)$, so we can assume that $p \mid r$. Write $r = pr_0$. Then $p^2r_0^2 - 4mn = p^2D_0$ implies $p^2 \mid 4mn$, and $(p, 2m) = 1$ shows $p^2 \mid n$, i.e. $n = p^2n_0$.

We leave the case $p = 2$ to the reader. \square

If $\phi \in J_{k,m}(\Gamma_0(N^2) \times (N\mathbb{Z} \times \mathbb{Z}), \chi)$ is a Jacobi form with integral algebraic coefficients which is an eigenfunction of all Hecke operators T_p for $(p, mN) = 1$ with corresponding eigenvalues λ_p , we define the set

$$\begin{aligned} A(p, \lambda_p) = & \{ \ell \text{ prime} : v_\ell \left(\lambda_p - \chi(p)\varepsilon \left(p^{k-1} + p^{k-2} \right) \right) > 0 \text{ for some } \varepsilon \in \{\pm 1\} \} \\ & \cup \{ \ell \text{ prime} : \ell \mid p(p-1) \}. \end{aligned} \quad (5)$$

The following result is a more precise variant of Theorem 1.1 from the introduction.

Theorem 3.5. *Let $\phi \in J_{k,m}(\Gamma_0(N^2) \times (N\mathbb{Z} \times \mathbb{Z}), \chi)$ be a Jacobi form with integral algebraic coefficients which is an eigenfunction of all Hecke operators T_p for $(p, mN) = 1$. Let $S = \{p_1, \dots, p_s\}$ be a non-empty finite set of odd primes which are coprime to mN and let $\varepsilon_1, \dots, \varepsilon_s \in \{\pm 1\}$. Let ℓ be a prime with $(\ell, 2mN) = 1$, $\ell \notin A(p_j, \lambda_{p_j})$ for all $p_j \in S$*

and $v_\ell(c(n^*, r^*)) = 0$ for some $D^* = r^{*2} - 4mn^* < 0$ with $(D^*, mN) = 1$. Then there exist infinitely many fundamental discriminants $D = r^2 - 4mn < 0$ prime to mN such that $\left(\frac{D}{p_j}\right) = \varepsilon_j$ for all $p_j \in S$ and $v_\ell(c(n, r)) = 0$.

Proof. By Lemma 2.4 the projected function

$$\phi' = \phi \prod_{p|mN} (1 - B_p) = \sum_{\substack{n, r \in \mathbb{Z} \\ r^2 < 4mn \\ (r, mN) = 1}} c(n, r) q^n \zeta^r$$

is a Jacobi form for $\Gamma_0(m^2 N^2) \times (mN\mathbb{Z} \times \mathbb{Z})$ with integral algebraic Fourier coefficients, and we have $v_\ell(\phi') = 0$ by the assumption on D^* . Further, ϕ' is still an eigenfunction of all Hecke operators T_p with $(p, mN) = 1$.

A repeated application of Proposition 3.1 shows that the function

$$\phi'' = \phi' \prod_{j=1}^{s-1} (1 - B_{p_j} + \varepsilon_j(\cdot) \psi_j) = \sum_{\substack{n, r \in \mathbb{Z} \\ (r, mN) = 1 \\ \left(\frac{D}{p_j}\right) = \varepsilon_j, 1 \leq j \leq s-1}} c(n, r) q^n \zeta^r$$

(with $\psi_j = \left(\frac{\cdot}{p_j}\right)$) again satisfies $v_\ell(\phi'') = 0$ and is an eigenform of T_p for p prime to mN and p_1, \dots, p_{s-1} .

If we now apply Proposition 3.2 to ϕ'' with $p = p_s$ and $\varepsilon = -\varepsilon_s$, we find infinitely many discriminants $D = r^2 - 4mn < 0$ prime to mN with distinct square free parts such that $\left(\frac{D}{p_j}\right) = \varepsilon_j$ for all $j = 1, \dots, s$ and $v_\ell(c(n, r)) = 0$. Writing each of these discriminants as $D = f^2 D_0$ with necessarily pairwise distinct fundamental discriminants D_0 , Lemma 3.4 shows that $D_0 = r_0^2 - 4mn_0$ is a square mod $4m$ with $c(n, r) = c(f^2 n_0, fr_0)$, and the eigenform relations from Lemma 3.3 give $v_\ell(c(n_0, r_0)) \leq v_\ell(c(n, r)) = 0$, i.e. $v_\ell(c(n_0, r_0)) = 0$. This completes the proof. \square

Now we restrict to $N = 1$. In order to infer Theorem 1.1 from the last theorem, it only remains to show that for a Hecke eigenform $\phi \in J_{k,m}$ (with $k > 2$ if ϕ is not a cusp form) and some p with $(p, 2m) = 1$ the set $A(p, \lambda_p)$ is finite, or equivalently, $\lambda_p \neq \pm(p^{k-1} + p^{k-2})$.

First suppose that $\phi \in J_{k,m}^{\text{cusp}}$ is a cusp form. By [15], Theorem 5, $J_{k,m}^{\text{cusp}}$ is Hecke-equivariantly isomorphic to a certain subspace of $S_{2k-2}(m)$, so λ_p is also an eigenvalue of the Hecke operator T_p on $S_{2k-2}(m)$. Now [8] gives the estimate $|\lambda_p| < p^{k-1} + p^{k-2}$.

If $0 \neq \phi \in J_{k,m}$ is a non-cuspidal Jacobi form of weight $k > 2$ and an eigenform of all Hecke operators T_p for $(p, m) = 1$, then by Theorem 4.4 in [5] the form ϕ lies in the space spanned by the $(U_d \circ V_{d'})$ -images of Eisenstein series $E_{k,m'}^{(\chi)}$ of index $m' = f^2$ defined after Theorem 2.4 in [5], where χ runs through the primitive Dirichlet characters mod f with $\chi(-1) = (-1)^k$. It was shown at the end of the proof of Theorem 2 in [15] that $E_{k,m'}^{(\chi)}$ is an eigenform of T_p for $(p, m') = 1$ with eigenvalue $\sigma_{2k-3,\chi}(p) = \sum_{d|p} d^{2k-3} \bar{\chi}(d) \chi(p/d)$. Hence the eigenvalues of ϕ are also of this form, and for $k > 2$ we have $\sigma_{2k-3,\chi}(p) \neq \pm(p^{k-1} + p^{k-2})$.

Thus the finite set of primes ℓ with $(\ell, 2m) = 1$ for which Theorem 1.1 does not hold is given by the primes in

$$\bigcap_{\substack{p \text{ prime} \\ (p, 2m)=1}} A(p, \lambda_p) \cup \{\ell \text{ prime} : v_\ell(c(n, r)) > 0 \text{ for all } r, n \text{ s.t. } (r, m) = 1\}. \quad (6)$$

Now let $\phi \in J_{k,m}^{\text{new}}$ be a newform in the sense of [5] which is an eigenform of all Hecke operators T_p with $(p, m) = 1$. Then ϕ is also an eigenform of the Atkin-Lehner involutions $w_{m'}$ for exact divisors m' of m , see [5], Theorem 5.2. The following proposition shows that if ϕ is a newform, we can replace in Theorem 3.5 the requirement that there be a discriminant $D^* = r^{*2} - 4mn^* < 0$ prime to m with $v_\ell(c(n^*, r^*)) = 0$ by a condition on the Atkin-Lehner eigenvalues of ϕ .

Proposition 3.6. *Let $\phi \in J_{k,m}^{\text{new}}$ be a Jacobi newform with integral algebraic coefficients which is an eigenfunction of all Hecke operators T_p for $(p, m) = 1$. Let ℓ be a prime with $(\ell, 2m) = 1$ and $v_\ell(\phi) = 0$ such that ℓ does not divide $p - \epsilon_p$ for any prime $p \parallel m$, where $\epsilon_p \in \{\pm 1\}$ denotes the eigenvalue of ϕ under the Atkin-Lehner involution w_p . Then there exists a discriminant $D^* = r^{*2} - 4mn^* < 0$ prime to m such that $v_\ell(c(n^*, r^*)) = 0$.*

Proof. The proof goes along the same lines as the proof of Lemma 3.1 in [15]: Assume that $v_\ell(c(n, r)) > 0$ for all $r^2 < 4mn$ with $(r, m) = 1$. Using the formula $\sum_{t|m} \mu(m) \prod_{p|t} e(r/p) = 0$ for $(r, m) > 1$ the assumption gives

$$v_\ell \left(\sum_{t|m} \mu(t) \prod_{p|t} \phi \Big|_m \left[0, \frac{g}{p} \right] \right) > 0$$

for every integer g . Let $f(\tau, z)$ denote the function in the braces. It is easily checked that f is a Jacobi form for the group $\Gamma_1(m^2) \ltimes (m\mathbb{Z} \times \mathbb{Z})$ with integral algebraic Fourier coefficients. Theorem 1.2 shows that for $M \in SL_2(\mathbb{Z})$ the function $f|_{k,m} M$ has algebraic Fourier coefficients and $v_\ell(f|_{k,m} M) \geq v_\ell(f) > 0$. Now the calculation in the proof of Lemma 3.1 in [15] gives

$$v_\ell \left(\phi \Big| \left(\prod_{d^2|m} \frac{\mu(d)}{d} u_d U_d \right) \circ \left(\prod_{p|m} \left(1 - \frac{1}{p} w_p \right) \right) \right) > 0$$

where u_d is the auxiliary operator from [15], line (6), and $\phi|U_d(\tau, z) = \phi(\tau, dz)$. Since ϕ is a newform and an eigenform of all Hecke operators, we have $\phi|u_d = 0$ for all $d^2|m$ with $d > 1$. In particular, we find

$$v_\ell \left(\phi \Big| \prod_{p|m} \left(1 - \frac{1}{p} w_p \right) \right) > 0.$$

The inner function is just $\prod_{p|m} (1 - \epsilon_p/p) \phi$, so we get $v_\ell(\phi) > 0$, contradicting the assumption $v_\ell(\phi) = 0$. Hence there is some $D^* = r^{*2} - 4mn^* < 0$ prime to m with $v_\ell(c(n^*, r^*)) = 0$. \square

Competing interests

The author declares that he has no competing interests.

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